

Form factors of local operators in a one-dimensional two-component Bose gas

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Abstract

We consider a one-dimensional model of a two-component Bose gas and study form factors of local operators in this model. For this aim we use an approach based on the algebraic Bethe ansatz. We show that the form factors under consideration can be reduced to those of the monodromy matrix entries in a generalized $GL(3)$ -invariant model. In this way we derive determinant representations for the form factors of local operators.

1 Introduction

In this paper we consider a one-dimensional model of Two-Component Bose Gas with δ -function repulsive interaction (TCBG model). This model is a generalization of the Lieb–Liniger model [1, 2], in which Bose fields have two internal degrees of freedom. The version with one internal degree of freedom is also known under different denominations: the Quantum nonlinear Schrödinger equation, Tonks–Girardeau gas [3, 4] or Gross–Pitaevskii model [5, 6], so that one

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can say that we are studying a two-component version of these models. The TCBG model was solved by C. N. Yang [7] who have found the eigenvectors and the spectrum of the Hamiltonian. The general approach to the solution of the model with n internal degrees of freedom (multi-component Bose gas) was given in [8] (see also [9, 10]). The nested algebraic Bethe ansatz was applied to this model in [11, 12].

We consider the TCBG model on a finite interval $[0, L]$ with periodic boundary conditions. The Hamiltonian of the model has the form

$$H = \int_0^L \left(\partial_x \Psi_\alpha^\dagger \partial_x \Psi_\alpha + \varkappa \Psi_\alpha^\dagger \Psi_\beta^\dagger \Psi_\beta \Psi_\alpha \right) dx, \quad (1.1)$$

where $\varkappa > 0$ is a coupling constant, $\alpha, \beta = 1, 2$ and the summation over repeated subscripts is assumed. The Bose fields $\Psi_\alpha(x)$ and $\Psi_\alpha^\dagger(x)$ satisfy canonical commutation relations

$$[\Psi_\alpha(x), \Psi_\beta^\dagger(y)] = \delta_{\alpha\beta} \delta(x - y). \quad (1.2)$$

The basis in the Fock space of the model is constructed by acting with operators $\Psi_\alpha^\dagger(x)$ onto the Fock vacuum $|0\rangle$ defined as

$$\Psi_\alpha(x)|0\rangle = 0, \quad \langle 0|\Psi_\alpha^\dagger(x) = 0, \quad \langle 0|0\rangle = 1. \quad (1.3)$$

Form factors of local operators of the TCBG model were studied in [13]. There, determinant representations for the form factors were obtained in some particular cases of the Hamiltonian eigenstates. In the present paper we consider a general case of form factors and obtain determinant representations for them. Our approach is based on recent results obtained in [14]. There we developed a method of calculating form factors of local operators in a wide class of $GL(3)$ -invariant integrable models solvable by the nested algebraic Bethe ansatz. However the method developed in [14] can be applied to the TCBG model only partly. Calculation of some form factors needs certain modifications. We consider these questions in the present paper.

In the models considered in [14] form factors of all monodromy matrix elements can be obtained by the zero modes method [15] from an initial form factor corresponding to one of the monodromy matrix entries. In its turn, the initial form factor should be calculated straightforwardly. The calculation is based on a sum formula for the scalar product of Bethe vectors [16] and a summation identity (see (4.2)). However the choice of the initial form factor is not fixed and we can use this freedom to deduce new summation identities. In [14] we have chosen as the initial form factor the one of a diagonal element of the monodromy matrix. Making another choice for this initial form factor, one can redo the full process through minor modifications of our calculations. Then, comparing the two final results, we obtain new summation identities. The latter can be directly used for the calculating form factors in the TCBG model.

The paper is organized as follows. In section 2 we describe the general settings of the algebraic Bethe ansatz. In particular, we introduce a composite model and give definitions of the zero modes in the TCBG model. In section 3 we formulate the main results of the paper and partly prove them. In particular we show that the determinant representations for the operators $\Psi_i^\dagger(x)\Psi_j(x)$ directly follow from the results of [14]. We also show that the form factors of Bose fields are related one to each other by simple transformations. In section 4.1 we derive a determinant representation for the form factor of the field $\Psi_2(x)$. In appendix we prove a summation identity that we use in section 4.1.

2 General scheme of the algebraic Bethe ansatz

In this section we describe a general scheme of the nested algebraic Bethe ansatz for $GL(3)$ -invariant models. We also point out a specification of some parameters in the case of the TCBG model.

The $GL(3)$ -invariant models are described by the following R -matrix acting on a space $V_1 \otimes V_2$, with $V_j = \mathbb{C}^3$

$$R(u, v) = \mathbf{I} + g(u, v)\mathbf{P}, \quad g(u, v) = \frac{c}{u - v}. \quad (2.1)$$

Here \mathbf{I} is the identity matrix in $V_1 \otimes V_2$, \mathbf{P} is the permutation matrix that exchanges V_1 and V_2 . The parameter c is related to the coupling constant of the TCBG model by $c = -i\kappa$.

The monodromy matrix $T(u)$ satisfies a standard RTT -relation

$$R_{12}(u, v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u, v). \quad (2.2)$$

The monodromy matrix $T(u)$ acts in $\mathbb{C}^3 \otimes \mathcal{H}$. In the TCBG model \mathcal{H} is the Fock space of the Hamiltonian (1.1). Equation (2.2) holds in the tensor product $V_1 \otimes V_2 \otimes \mathcal{H}$, and the matrices $T_k(w)$ act non-trivially in $V_k \otimes \mathcal{H}$. The Fock vacuum vector $|0\rangle$ is annihilated by the operators $T_{ij}(w)$ with $i > j$. The dual vector $\langle 0|$ is annihilated by the operators $T_{ij}(w)$ with $i < j$. Both vectors are eigenvectors of the diagonal entries of the monodromy matrix

$$T_{kk}(w)|0\rangle = r_k(w)|0\rangle, \quad \langle 0|T_{kk}(w) = r_k(w)\langle 0|, \quad k = 1, 2, 3. \quad (2.3)$$

Without loss of generality we assume that $r_2(w) = 1$. In the TCBG model we also have $r_1(w) = 1$ and $r_3(w) = e^{i\omega L}$. However, up to a certain point it is convenient not to use explicit expressions for the functions $r_k(w)$. Therefore we shall continue to use the notation $r_k(w)$, making its specification if necessary.

Bethe vectors are certain polynomials in operators $T_{ij}(u)$ with $i < j$ acting on the vector $|0\rangle$ [17–21]. In the $GL(3)$ -invariant models they depend on two sets of variables called Bethe parameters. We denote the Bethe vectors $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$. Here the Bethe parameters are $\bar{u} = \{u_1, \dots, u_a\}$ and $\bar{v} = \{v_1, \dots, v_b\}$. We call them u -type variables and v -type variables. The subscripts a and b ($a, b = 0, 1, \dots$) respectively denote the cardinalities of the sets \bar{u} and \bar{v} . The peculiarity of the TCBG model is that the Bethe vectors do not depend on the operators $T_{12}(u)$: they are polynomials in $T_{13}(v)$ and $T_{23}(v)$ only. One more restriction for the Bethe vectors of the TCBG model is that $a \leq b$. However these conditions do not play an essential role in our considerations.

Similarly we can construct dual Bethe vectors in the dual space as polynomials in operators $T_{ij}(u)$ with $i > j$ acting on $\langle 0|$. We denote them $\mathbb{C}_{a,b}(\bar{u}; \bar{v})$ with the same meaning for the arguments and subscripts. Dual Bethe vectors of the TCBG model exist for $a \leq b$ and they do not depend on $T_{21}(u)$.

2.1 Notation

Besides the function $g(u, v)$ we also introduce a function $f(u, v)$

$$f(u, v) = \frac{u - v + c}{u - v}. \quad (2.4)$$

We denote sets of variables by bar: \bar{u} , \bar{v} etc. If necessary, the cardinalities of the sets are given in special comments. Individual elements of the sets are denoted by subscripts: w_j , u_k etc. We say that $\bar{x} = \bar{x}'$, if $\#\bar{x} = \#\bar{x}'$ and $x_i = x'_i$ (up to a permutation) for $i = 1, \dots, \#\bar{x}$. We say that $\bar{x} \neq \bar{x}'$ otherwise.

Below we consider partitions of sets into subsets. The notation $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_{II}\}$ means that the set \bar{u} is divided into two disjoint subsets \bar{u}_I and \bar{u}_{II} . As a rule, we use roman numbers for subscripts of subsets: \bar{u}_I , \bar{v}_{II} etc. However, if we deal with a big quantity of subsets, then we use standard arabic numbers for their notation. In such cases we explicitly indicate it.

Similarly to the paper [22] we use a shorthand notation for products of some functions. Namely, if the functions r_i (2.3) or the function f (2.4) depend on sets of variables, this means that one should take the product over the corresponding set. For example,

$$r_1(\bar{u}) = \prod_{u_k \in \bar{u}} r_1(u_k); \quad f(z, \bar{w}) = \prod_{w_j \in \bar{w}} f(z, w_j); \quad f(\bar{v}_{II}, \bar{u}_I) = \prod_{u_j \in \bar{u}_I} \prod_{v_k \in \bar{v}_{II}} f(v_k, u_j). \quad (2.5)$$

In the last equation it is assumed that the sets \bar{u} and \bar{v} are divided into several subsets and the product is taken over the subsets \bar{v}_{II} and \bar{u}_I . By definition any product with respect to the empty set is equal to 1. If we have a double product, then it is also equal to 1 if at least one of sets is empty.

In section 2.4 we shall introduce several new scalar functions and will extend the convention (2.5) to their products.

2.2 On-shell Bethe vectors

In the algebraic Bethe ansatz the role of a quantum Hamiltonian is played by the transfer matrix. It is the trace in the auxiliary space of the monodromy matrix: $\text{tr} T(u)$. The eigenstates of the transfer matrix are called *on-shell* Bethe vectors. The eigenstates of the transfer matrix in the dual space are called dual on-shell Bethe vectors².

A (dual) Bethe vector becomes on-shell, if the Bethe parameters satisfy the system of Bethe equations. We give this system in a slightly unusual form

$$r_1(\bar{u}_I) = \frac{f(\bar{u}_I, \bar{u}_{II})}{f(\bar{u}_{II}, \bar{u}_I)} f(\bar{v}, \bar{u}_I), \quad r_3(\bar{v}_I) = \frac{f(\bar{v}_{II}, \bar{v}_I)}{f(\bar{v}_I, \bar{v}_{II})} f(\bar{v}_I, \bar{u}). \quad (2.6)$$

These equations should hold for arbitrary partitions of the sets \bar{u} and \bar{v} into subsets $\{\bar{u}_I, \bar{u}_{II}\}$ and $\{\bar{v}_I, \bar{v}_{II}\}$ with $\#\bar{u}_I = \#\bar{v}_I = 1$. It is easy to see that if the sets \bar{u} and \bar{v} satisfy the system (2.6), then they satisfy the same system without the restriction $\#\bar{u}_I = \#\bar{v}_I = 1$. In particular, if $\bar{u}_{II} = \bar{v}_{II} = \emptyset$, we obtain

$$r_1(\bar{u}) = r_3(\bar{v}) = f(\bar{v}, \bar{u}) = 1, \quad (2.7)$$

because $r_1(u) = 1$ in the TCBG model.

If the sets \bar{u} and \bar{v} satisfy (2.6), then

$$\text{tr} T(w) \mathbb{B}_{a,b}(\bar{u}, \bar{v}) = \tau(w|\bar{u}, \bar{v}) \mathbb{B}_{a,b}(\bar{u}, \bar{v}), \quad \mathbb{C}_{a,b}(\bar{u}, \bar{v}) \text{tr} T(w) = \tau(w|\bar{u}, \bar{v}) \mathbb{C}_{a,b}(\bar{u}, \bar{v}), \quad (2.8)$$

²For simplicity here and below we do not distinguish between vectors and dual vectors, because their properties are completely analogous to each other.

with

$$\tau(w|\bar{u}, \bar{v}) = r_1(w)f(\bar{u}, w) + f(w, \bar{u})f(\bar{v}, w) + r_3(w)f(w, \bar{v}). \quad (2.9)$$

2.3 Scalar products and form factors

Scalar product of generic Bethe vectors is defined as follows:

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}_{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B). \quad (2.10)$$

An expression for the scalar products in terms of a sum over partitions of Bethe parameters (sum formula) was found in [16]

$$\begin{aligned} \mathcal{S}_{a,b} = & \sum r_1(\bar{u}_I^B) r_1(\bar{u}_I^C) r_3(\bar{v}_I^C) r_3(\bar{v}_I^B) f(\bar{u}_I^C, \bar{u}_I^B) f(\bar{u}_I^B, \bar{u}_I^C) \\ & \times f(\bar{v}_I^C, \bar{v}_I^B) f(\bar{v}_I^B, \bar{v}_I^C) \frac{f(\bar{v}_I^C, \bar{u}_I^C) f(\bar{v}_I^B, \bar{u}_I^B)}{f(\bar{v}_I^C, \bar{u}_I^C) f(\bar{v}_I^B, \bar{u}_I^B)} Z_{a_I, b_I}(\bar{u}_I^C; \bar{u}_I^B | \bar{v}_I^C; \bar{v}_I^B) Z_{a_I, b_I}(\bar{u}_I^B; \bar{u}_I^C | \bar{v}_I^B; \bar{v}_I^C). \end{aligned} \quad (2.11)$$

Here all the Bethe parameters are generic complex numbers and the sum is taken over the partitions of the sets \bar{u}^C , \bar{u}^B , \bar{v}^C , and \bar{v}^B

$$\begin{aligned} \bar{u}^C &\Rightarrow \{\bar{u}_I^C, \bar{u}_I^B\}, & \bar{v}^C &\Rightarrow \{\bar{v}_I^C, \bar{v}_I^B\}, \\ \bar{u}^B &\Rightarrow \{\bar{u}_I^B, \bar{u}_I^C\}, & \bar{v}^B &\Rightarrow \{\bar{v}_I^B, \bar{v}_I^C\}. \end{aligned} \quad (2.12)$$

The partitions are independent except $\#\bar{u}_I^C = \#\bar{u}_I^B = a_I$ and $\#\bar{v}_I^C = \#\bar{v}_I^B = b_I$. Accordingly one has $\#\bar{u}_I^C = \#\bar{u}_I^B = a_I = a - a_I$ and $\#\bar{v}_I^C = \#\bar{v}_I^B = b_I = b - b_I$.

The rational functions Z_{a_I, b_I} and Z_{a_I, b_I} are the so-called highest coefficients. They are equal to a partition function of 15-vertex model with special boundary conditions [16]. The reader can find their explicit representations in [23, 24]. We do not use these explicit formulas in the present paper except $Z_{0,0}(\emptyset; \emptyset | \emptyset; \emptyset) = 1$. This condition is needed to satisfy the normalization $\mathcal{S}_{0,0} = \langle 0|0 \rangle = 1$ (see (1.3)). Note also that the subscripts of the highest coefficient are equal to the cardinalities of the subsets to the left and to the right of the vertical line.

Form factors of the monodromy matrix entries are defined as

$$\mathcal{F}_{a,b}^{(i,j)}(z) \equiv \mathcal{F}_{a,b}^{(i,j)}(z | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}_{a',b'}(\bar{u}^C; \bar{v}^C) T_{ij}(z) \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B), \quad (2.13)$$

where both $\mathbb{C}_{a',b'}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B)$ are on-shell Bethe vectors, and

$$\begin{aligned} a' &= a + \delta_{i1} - \delta_{j1}, \\ b' &= b + \delta_{j3} - \delta_{i3}. \end{aligned} \quad (2.14)$$

The parameter z is an arbitrary complex number. We call it the external parameter.

It was proved in [15] that if $\{\bar{u}^C, \bar{v}^C\} \neq \{\bar{u}^B, \bar{v}^B\}$, then the combination

$$\mathfrak{F}_{a,b}^{(i,j)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \frac{\mathcal{F}_{a,b}^{(i,j)}(z | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)}{\tau(z | \bar{u}^C, \bar{v}^C) - \tau(z | \bar{u}^B, \bar{v}^B)} \quad (2.15)$$

does not depend on z . We call $\mathfrak{F}_{a,b}^{(i,j)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)$ the *universal form factor* of the operator $T_{ij}(z)$. If $\bar{u}^C \cap \bar{u}^B = \emptyset$ and $\bar{v}^C \cap \bar{v}^B = \emptyset$, then the universal form factor is determined by the R -matrix only. It does not depend on a specific model, in particular, on the functions $r_1(z)$ and $r_3(z)$. Determinant representations for $\mathfrak{F}_{a,b}^{(i,j)}$ were obtained in [15, 25–28].

Due to the invariance of the R -matrix under transposition with respect to both spaces, the mapping

$$\psi : T_{ij}(u) \mapsto T_{ji}(u) \quad (2.16)$$

defines an antimorphism of the algebra (2.2). The mapping (2.16) acts in the algebra (2.2), therefore expectation values of the operators $T_{ij}(z)$ are invariant under the action of ψ . In particular,

$$\psi \left(\mathcal{F}_{a,b}^{(i,j)}(z | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) \right) = \mathcal{F}_{a,b}^{(i,j)}(z | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B). \quad (2.17)$$

On the other hand, we have

$$\psi \left(\mathbb{C}_{a',b'}(\bar{u}^C; \bar{v}^C) T_{ij}(z) \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B) \right) = \mathbb{C}_{a,b}(\bar{u}^B; \bar{v}^B) T_{ji}(z) \mathbb{B}_{a',b'}(\bar{u}^C; \bar{v}^C), \quad (2.18)$$

and we recognize the form factor of the operator T_{ji} in the r.h.s. Thus, we obtain simple relations between different form factors:

$$\begin{aligned} \mathcal{F}_{a,b}^{(i,j)}(z | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) &= \mathcal{F}_{a',b'}^{(j,i)}(z | \bar{u}^B, \bar{v}^B; \bar{u}^C, \bar{v}^C), \\ \mathfrak{F}_{a,b}^{(i,j)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) &= -\mathfrak{F}_{a',b'}^{(j,i)}(\bar{u}^B, \bar{v}^B; \bar{u}^C, \bar{v}^C). \end{aligned} \quad (2.19)$$

In the TCBG model the antimorphism ψ implies the following mapping of the Bose fields [29]:

$$\psi(\Psi_i(x)) = -\Psi_i^\dagger(L-x), \quad \psi(\Psi_i^\dagger(x)) = -\Psi_i(L-x). \quad (2.20)$$

Due to (2.20) we can relate form factors of the fields $\Psi_i(L-x)$ and $\Psi_i^\dagger(x)$

$$\psi \left(\mathbb{C}_{a',b'}(\bar{u}^C; \bar{v}^C) \Psi_i(L-x) \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B) \right) = -\mathbb{C}_{a,b}(\bar{u}^B; \bar{v}^B) \Psi_i^\dagger(x) \mathbb{B}_{a',b'}(\bar{u}^C; \bar{v}^C). \quad (2.21)$$

Thus, it is enough to calculate the form factors of the fields $\Psi_i(x)$. The form factors of $\Psi_i^\dagger(x)$ can be obtained from the latter via (2.21).

2.4 Composite model

In the composite model the total monodromy matrix $T(u)$ is presented as a usual matrix product of the partial monodromy matrices $T^{(2)}(u)$ and $T^{(1)}(u)$:

$$T(u) = T^{(2)}(u) T^{(1)}(u). \quad (2.22)$$

The matrix elements of $T(u)$ are operators in the space of states \mathcal{H} that corresponds to an interval $[0, L]$. The matrix elements of the partial monodromy matrices $T^{(1)}(u)$ and $T^{(2)}(u)$ act in the spaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ corresponding to the intervals $[0, x]$ and $[x, L]$ respectively. Here x is an intermediate point of the interval $[0, L]$. The total space of states \mathcal{H} is a tensor product of

the partial spaces of states $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$. The operators $T_{ij}^{(2)}(u)$ and $T_{kl}^{(1)}(v)$ commute one with each other, as they act in different spaces.

Every $T^{(l)}(u)$ satisfies RTT -relation (2.2) and has its own vacuum state $|0\rangle^{(l)}$ and a dual state $\langle 0|^{(l)}$. Hereby $|0\rangle = |0\rangle^{(1)} \otimes |0\rangle^{(2)}$ and $\langle 0| = \langle 0|^{(1)} \otimes \langle 0|^{(2)}$.

The properties of the partial vacuum vectors are similar to their total analogs, in particular,

$$T_{kk}^{(l)}(u)|0\rangle^{(l)} = r_k^{(l)}(u)|0\rangle^{(l)}, \quad \langle 0|^{(l)}T_{kk}^{(l)}(u) = r_k^{(l)}(u)\langle 0|^{(l)}, \quad l = 1, 2, \quad (2.23)$$

where $r_k^{(l)}(u)$ are some complex valued functions for $k = 1, 3$ and $r_2^{(l)}(u) = 1$. In the TCBG model we have $r_1^{(l)}(u) = 1$, $r_3^{(1)}(u) = e^{iux}$, and $r_3^{(2)}(u) = e^{iu(L-x)}$. Evidently

$$r_k(u) = r_k^{(1)}(u)r_k^{(2)}(u). \quad (2.24)$$

Below we express form factors in terms of $r_k^{(1)}(u)$, therefore we introduce a special notation for these functions

$$r_k^{(1)}(u) = \ell_k(u), \quad \text{and hence,} \quad r_k^{(2)}(u) = \frac{r_k(u)}{\ell_k(u)}, \quad k = 1, 3. \quad (2.25)$$

Thus, in the TCBG model $\ell_1(u) = 1$ and $\ell_3(u) = e^{iux}$, however, up to a certain point we continue to use the notation $\ell_k(u)$ without its specification.

We extend convention (2.5) to the products of the functions $r_k^{(l)}(u)$ and $\ell_k(u)$. Namely, whenever these functions depend on a set of variables this means the product over the corresponding set.

Finally, we recall the formulas for total on-shell (dual) Bethe vector in terms of partial (dual) Bethe vectors. They have the form [18, 22]

$$\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \sum \frac{\ell_3(\bar{v}_{\text{II}})}{\ell_1(\bar{u}_{\text{I}})} f(\bar{u}_{\text{I}}, \bar{u}_{\text{II}}) f(\bar{v}_{\text{II}}, \bar{v}_{\text{I}}) f(\bar{v}_{\text{I}}, \bar{u}_{\text{I}}) \mathbb{B}_{a_{\text{I}}, b_{\text{I}}}^{(1)}(\bar{u}_{\text{I}}; \bar{v}_{\text{I}}) \mathbb{B}_{a_{\text{II}}, b_{\text{II}}}^{(2)}(\bar{u}_{\text{II}}; \bar{v}_{\text{II}}), \quad (2.26)$$

and

$$\mathbb{C}_{a,b}(\bar{u}; \bar{v}) = \sum \frac{\ell_1(\bar{u}_{\text{II}})}{\ell_3(\bar{v}_{\text{I}})} f(\bar{u}_{\text{I}}, \bar{u}_{\text{II}}) f(\bar{v}_{\text{II}}, \bar{v}_{\text{I}}) f(\bar{v}_{\text{I}}, \bar{u}_{\text{I}}) \mathbb{C}_{a_{\text{I}}, b_{\text{I}}}^{(1)}(\bar{u}_{\text{I}}; \bar{v}_{\text{I}}) \mathbb{C}_{a_{\text{II}}, b_{\text{II}}}^{(2)}(\bar{u}_{\text{II}}; \bar{v}_{\text{II}}). \quad (2.27)$$

In (2.26) $\mathbb{B}_{a,b}$ is an on-shell Bethe vector of the total monodromy matrix $T(u)$, while $\mathbb{B}_{a_{\text{I}}, b_{\text{I}}}^{(l)}$ are Bethe vectors of the partial monodromy matrices $T^{(l)}(u)$ (partial Bethe vectors). Similarly equation (2.27) expresses a dual total on-shell Bethe vectors in terms of partial dual Bethe vectors. In both formulas the sums are taken over all possible partitions $\bar{u} \Rightarrow \{\bar{u}_{\text{I}}, \bar{u}_{\text{II}}\}$ and $\bar{v} \Rightarrow \{\bar{v}_{\text{I}}, \bar{v}_{\text{II}}\}$. The cardinalities of the subsets are shown by the subscripts of (dual) partial Bethe vectors and they run through all possible values.

2.5 Total and partial zero modes

The most principal difference between the TCBG model and the models considered in [14] appears in the definition of the monodromy matrix zero modes. It was assumed in [14] that the

monodromy matrix $T(u)$ goes to the identity operator at $|u| \rightarrow \infty$. In the TCBG model this is true only for the left-upper 2×2 block of $T(u)$. The properties of the zero modes in the TCBG model and their relations to the Bose fields were found in [29]. In this section we list several results necessary for further applications.

We consider the zero modes of the total monodromy matrix $T(u)$ and partial zero modes of the partial monodromy matrices $T^{(l)}(u)$ (mainly for $l = 1$). For $i, j = 1, 2$ the matrix elements $T_{ij}(u)$ and $T_{ij}^{(l)}(u)$ have the following asymptotic expansions:

$$\begin{aligned} T_{ij}(u) &= \delta_{ij} + \sum_{n=0}^{\infty} T_{ij}[n] \left(\frac{c}{u}\right)^{n+1}, \\ T_{ij}^{(l)}(u) &= \delta_{ij} + \sum_{n=0}^{\infty} T_{ij}^{(l)}[n] \left(\frac{c}{u}\right)^{n+1}, \quad l = 1, 2, \end{aligned} \quad |u| \rightarrow \infty. \quad (2.28)$$

Accordingly the total and partial zero modes are defined as

$$\begin{aligned} T_{ij}[0] &= \lim_{|u| \rightarrow \infty} \frac{u}{c} (T_{ij}(u) - \delta_{ij}), \\ T_{ij}^{(l)}[0] &= \lim_{|u| \rightarrow \infty} \frac{u}{c} (T_{ij}^{(l)}(u) - \delta_{ij}), \quad l = 1, 2, \end{aligned} \quad i, j = 1, 2. \quad (2.29)$$

It is easy to see that $T_{ij}[0] = T_{ij}^{(1)}[0] + T_{ij}^{(2)}[0]$.

The partial zero modes $T_{ij}^{(1)}[0]$ ($i, j = 1, 2$) have the following explicit representation in terms of the Bose fields Ψ_i and Ψ_i^\dagger :

$$T_{ij}^{(1)}[0] = - \int_0^x \Psi_i^\dagger(y) \Psi_j(y) dy, \quad i, j = 1, 2. \quad (2.30)$$

Thus, computing form factors of these zero modes and taking the derivative over x we obtain form factors of local operators $\Psi_i^\dagger(x) \Psi_j(x)$.

The action of the total and partial zero modes $T_{ii}[0]$ and $T_{ii}^{(1)}[0]$ ($i = 1, 2$) onto the corresponding Bethe vectors has the following form

$$\begin{aligned} T_{11}[0] \mathbb{B}_{a,b}(\bar{u}; \bar{v}) &= -a \mathbb{B}_{a,b}(\bar{u}; \bar{v}), & T_{11}^{(1)}[0] \mathbb{B}_{a,b}^{(1)}(\bar{u}; \bar{v}) &= -a \mathbb{B}_{a,b}^{(1)}(\bar{u}; \bar{v}), \\ T_{22}[0] \mathbb{B}_{a,b}(\bar{u}; \bar{v}) &= (a - b) \mathbb{B}_{a,b}(\bar{u}; \bar{v}), & T_{22}^{(1)}[0] \mathbb{B}_{a,b}^{(1)}(\bar{u}; \bar{v}) &= (a - b) \mathbb{B}_{a,b}^{(1)}(\bar{u}; \bar{v}). \end{aligned} \quad (2.31)$$

In these formulas $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ and $\mathbb{B}_{a,b}^{(1)}(\bar{u}; \bar{v})$ respectively are generic total and partial Bethe vectors, i.e. their Bethe parameters are generic complex numbers.

The action of the zero modes $T_{12}[0]$ and $T_{21}[0]$ (and their partial analogs) is

$$\begin{aligned} T_{12}[0] \mathbb{B}_{a,b}(\bar{u}; \bar{v}) &= \lim_{|w| \rightarrow \infty} \frac{w}{c} \mathbb{B}_{a+1,b}(\{w, \bar{u}\}; \bar{v}), & T_{12}^{(1)}[0] \mathbb{B}_{a,b}^{(1)}(\bar{u}; \bar{v}) &= \lim_{|w| \rightarrow \infty} \frac{w}{c} \mathbb{B}_{a+1,b}^{(1)}(\{w, \bar{u}\}; \bar{v}), \\ \mathbb{C}_{a,b}(\bar{u}; \bar{v}) T_{21}[0] &= \lim_{|w| \rightarrow \infty} \frac{w}{c} \mathbb{C}_{a+1,b}(\{w, \bar{u}\}; \bar{v}), & \mathbb{C}_{a,b}^{(1)}(\bar{u}; \bar{v}) T_{21}^{(1)}[0] &= \lim_{|w| \rightarrow \infty} \frac{w}{c} \mathbb{C}_{a+1,b}^{(1)}(\{w, \bar{u}\}; \bar{v}). \end{aligned} \quad (2.32)$$

Here also the (dual) Bethe vectors are generic. It is important to note, however, that if the (dual) Bethe vectors are on-shell, then the resulting vectors also are on-shell, because the Bethe equations have infinite roots of u -type.

Finally we need singular properties of total on-shell (dual) Bethe vectors

$$\mathbb{C}_{a,b}(\bar{u}; \bar{v}) T_{12}[0] = 0, \quad T_{21}[0] \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = 0. \quad (2.33)$$

Here $\mathbb{C}_{a,b}(\bar{u}; \bar{v})$ and $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ are on-shell Bethe vectors. This property was found in [30] for $GL(N)$ -invariant models. In the $GL(3)$ case it follows from the explicit formulas of the action of the operators $T_{ij}(z)$ onto Bethe vectors [21].

The definitions of the (total and partial) zero modes of the operators T_{i3} and T_{3i} are different from (2.28). The operator $T_{33}(u)$ has the following expansion

$$\begin{aligned} T_{33}(u) &= e^{iLu} + e^{iLu} \sum_{n=0}^{\infty} T_{33}[n] \left(\frac{c}{u}\right)^{n+1}, \\ T_{33}^{(1)}(u) &= e^{ixu} + e^{ixu} \sum_{n=0}^{\infty} T_{33}^{(1)}[n] \left(\frac{c}{u}\right)^{n+1}. \end{aligned} \quad (2.34)$$

Respectively the total and partial zero modes are defined as

$$\begin{aligned} T_{33}[0] &= \lim_{|u| \rightarrow \infty} \frac{u}{c} (e^{-iLu} T_{33}(u) - 1), \\ T_{33}^{(1)}[0] &= \lim_{|u| \rightarrow \infty} \frac{u}{c} (e^{-ixu} T_{33}^{(1)}(u) - 1). \end{aligned} \quad (2.35)$$

Expansions (2.28), (2.35) imply the following expansion of the functions $\ell_k(u)$:

$$\begin{aligned} \ell_1(u) &= 1 + \ell_1[0] \frac{c}{u} + o(u^{-1}), \\ \ell_3(u) &= e^{ixu} \left(1 + \ell_3[0] \frac{c}{u} + o(u^{-1}) \right), \end{aligned} \quad |u| \rightarrow \infty. \quad (2.36)$$

Since $\ell_1(u) = 1$ and $\ell_3(u) = e^{ixu}$, we conclude that $\ell_1[0] = \ell_3[0] = 0$.

It turns out that in the TCBG model $T_{33}[0] = -T_{11}[0] - T_{22}[0]$ (and similarly for the partial zero modes), therefore below we do not consider these zero modes.

Asymptotic expansions of the operators T_{i3} and T_{3i} with $i = 1, 2$ are more sophisticated. We give them for the partial zero modes of the operators $T_{i3}^{(1)}(u)$ and $T_{3j}^{(1)}(u)$

$$\begin{aligned} T_{i3}^{(1)}(u) &= -\frac{\sqrt{\mathcal{X}}}{u} \left(e^{iux} \Psi_i^\dagger(x) - \Psi_i^\dagger(0) \right) + O(u^{-2}), \quad i = 1, 2, \\ T_{3j}^{(1)}(u) &= -\frac{\sqrt{\mathcal{X}}}{u} \left(\Psi_j(x) - e^{iux} \Psi_j(0) \right) + O(u^{-2}), \quad j = 1, 2. \end{aligned} \quad (2.37)$$

For the total zero modes one should replace x by L everywhere in these formulas. We see that the asymptotic behavior $|u| \rightarrow \infty$ leads to two types of zero modes, corresponding to the two boundaries of the interval $[0, x]$: the left (resp. right) boundary corresponds to the left partial

zero modes $T_{ij}^{(1;L)}[0]$ (resp. the right ones $T_{ij}^{(1;R)}[0]$). For our goal we need only the right partial zero modes, which are defined as follows:

$$\begin{aligned} T_{i3}^{(1;R)}[0] &= \lim_{u \rightarrow -i\infty} e^{-iux} \frac{u}{c} T_{i3}^{(1)}(u), \\ T_{3i}^{(1;R)}[0] &= \lim_{u \rightarrow +i\infty} \frac{u}{c} T_{3i}^{(1)}(u), \end{aligned} \quad i = 1, 2, \quad (2.38)$$

and hence,

$$T_{i3}^{(1;R)}[0] = \frac{1}{i\sqrt{\varkappa}} \Psi_i^\dagger(x), \quad T_{3i}^{(1;R)}[0] = \frac{1}{i\sqrt{\varkappa}} \Psi_i(x), \quad i = 1, 2. \quad (2.39)$$

Thus, calculating the form factors of these zero modes leads to the evaluation of the form factors of the local fields $\Psi_i(x)$ and $\Psi_i^\dagger(x)$.

Below we will need the actions of the right partial zero modes $T_{23}^{(1;R)}[0]$ and $T_{32}^{(1;R)}[0]$ respectively onto usual and dual partial Bethe vectors:

$$\begin{aligned} T_{23}^{(1;R)}[0] \mathbb{B}_{a,b}^{(1)}(\bar{u}; \bar{v}) &= \lim_{w \rightarrow -i\infty} e^{-iwx} \frac{w}{c} \mathbb{B}_{a,b+1}^{(1)}(\bar{u}; \{w, \bar{v}\}), \\ \mathbb{C}_{a,b}^{(1)}(\bar{u}; \bar{v}) T_{32}^{(1;R)}[0] &= \lim_{w \rightarrow +i\infty} \frac{w}{c} \mathbb{C}_{a,b+1}^{(1)}(\bar{u}; \{w, \bar{v}\}). \end{aligned} \quad (2.40)$$

Here both Bethe vectors are generic.

3 Main results

In this section we give a list of formulas for the form factors of local operators of the TCBG model in terms of the universal form factors (2.15). The reader can find determinant representations for the universal form factors in [15, 25–28].

For given on-shell vectors $\mathbb{C}_{a',b'}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}_{a,b}(\bar{u}^C; \bar{v}^C)$ define an excitation momentum as

$$\mathcal{P}(\bar{v}^B, \bar{v}^C) = \sum_{i=1}^b v_i^B - \sum_{i=1}^{b'} v_i^C. \quad (3.1)$$

Theorem 3.1. *Let $\mathbb{C}_{a',b'}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}_{a,b}(\bar{u}^C; \bar{v}^C)$ be on-shell Bethe vectors such that $\{\bar{u}^C, \bar{v}^C\} \neq \{\bar{u}^B, \bar{v}^B\}$. Then the form factors of the operators $\Psi_i^\dagger(x) \Psi_j(x)$ ($i, j = 1, 2$) have the following representation:*

$$\mathbb{C}_{a',b}(\bar{u}^C; \bar{v}^C) \Psi_i^\dagger(x) \Psi_j(x) \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B) = -i \mathcal{P}(\bar{v}^B, \bar{v}^C) e^{ix \mathcal{P}(\bar{v}^B, \bar{v}^C)} \mathfrak{F}_{a,b}^{(i,j)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B), \quad (3.2)$$

where $\mathfrak{F}_{a,b}^{(i,j)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)$ is the universal form factor of the matrix element $T_{ij}(z)$ and $a' = a + j - i$.

Theorem 3.1 is a direct corollary of determinant representations for the partial zero modes obtained in [14]

$$\mathbb{C}_{a',b'}(\bar{u}^C; \bar{v}^C) T_{ij}^{(1)}[0] \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B) = \left(\frac{\ell_1(\bar{u}^C) \ell_3(\bar{v}^B)}{\ell_1(\bar{u}^B) \ell_3(\bar{v}^C)} - 1 \right) \mathfrak{F}_{a,b}^{(i,j)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B), \quad (3.3)$$

where $a' = a + \delta_{i1} - \delta_{j1}$, $b' = b + \delta_{j3} - \delta_{i3}$. We have seen that for the matrix elements $T_{ij}^{(1)}(u)$ with $i, j = 1, 2$ the actions of the zero modes onto Bethe vectors in the TCBG model are the same as in the models considered in [14]. Therefore the form factors of the partial zero modes also are the same. One should only specify $\ell_1(u) = 1$, $\ell_3(v) = e^{ixv}$ in (3.3) and use (2.30).

Let $\bar{\kappa} = \{\kappa_1, \kappa_2, \kappa_3\}$. Consider the following deformation of the Bethe equations (2.6)

$$1 = \frac{\kappa_2}{\kappa_1} \frac{f(\bar{u}_I, \bar{u}_{II})}{f(\bar{u}_{II}, \bar{u}_I)} f(\bar{v}, \bar{u}_I), \quad r_3(\bar{v}_I) = \frac{\kappa_2}{\kappa_3} \frac{f(\bar{v}_{II}, \bar{v}_I)}{f(\bar{v}_I, \bar{v}_{II})} f(\bar{v}_I, \bar{u}), \quad (3.4)$$

where $\#\bar{u}_I = \#\bar{v}_I = 1$. This system is called twisted Bethe equations. It determines the roots v_i and u_i as implicit functions of the parameters $\bar{\kappa}$: $v_i = v_i(\bar{\kappa})$ and $u_i = u_i(\bar{\kappa})$.

Theorem 3.2. *Let $\{\bar{u}^C, \bar{v}^C\} = \{\bar{u}^B, \bar{v}^B\} = \{\bar{u}, \bar{v}\}$. Then the form factors of the operators $\Psi_j^\dagger(x)\Psi_j(x)$ ($j = 1, 2$) have the following representation:*

$$\mathbb{C}_{a,b}(\bar{u}; \bar{v}) \Psi_j^\dagger(x)\Psi_j(x) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = i \sum_{k=1}^b \frac{dv_k(\bar{\kappa})}{d\kappa_j} \Big|_{\bar{\kappa}=1} \cdot \|\mathbb{B}_{a,b}(\bar{u}; \bar{v})\|^2, \quad j = 1, 2, \quad (3.5)$$

$$\mathbb{C}_{a,b}(\bar{u}; \bar{v}) (\Psi_1^\dagger(x)\Psi_1(x) + \Psi_2^\dagger(x)\Psi_2(x)) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \frac{b}{L} \|\mathbb{B}_{a,b}(\bar{u}; \bar{v})\|^2, \quad (3.6)$$

where $\bar{v}(\bar{\kappa})$ is a deformation of \bar{v} , such that the set $\bar{v}(\bar{\kappa})$ satisfies twisted Bethe equations (3.4) and $\bar{v}(\bar{\kappa}) = \bar{v}$ at $\bar{\kappa} = \{1, 1, 1\}$.

Equation (3.5) of theorem 3.2 also directly follows from the corresponding representation for the expectation value of the partial zero modes obtained in [14]

$$\mathbb{C}_{a,b}(\bar{u}; \bar{v}) T_{ii}^{(1)}[0] \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \left(\delta_{i,1} \ell_1[0] + \delta_{i,3} \ell_3[0] + \frac{d}{d\kappa_i} \log \frac{\ell_1(\bar{u}(\bar{\kappa}))}{\ell_3(\bar{v}(\bar{\kappa}))} \Big|_{\bar{\kappa}=1} \right) \|\mathbb{B}_{a,b}(\bar{u}; \bar{v})\|^2. \quad (3.7)$$

Setting here $\ell_1[0] = \ell_3[0] = 0$, $\ell_1(u) = 1$, and $\ell_3(v) = e^{ixv}$ we immediately arrive at (3.5). Furthermore, for a special choice $\kappa_1 = \kappa_2 = \kappa$, $\kappa_3 = 1$ equation (3.7) takes the following form:

$$\mathbb{C}_{a,b}(\bar{u}; \bar{v}) (T_{11}^{(1)}[0] + T_{22}^{(1)}[0]) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \left(\ell_1[0] + \frac{d}{d\kappa} \log \frac{\ell_1(\bar{u}(\kappa))}{\ell_3(\bar{v}(\kappa))} \Big|_{\kappa=1} \right) \|\mathbb{B}_{a,b}(\bar{u}; \bar{v})\|^2. \quad (3.8)$$

It is easy to see that for this special choice of $\bar{\kappa}$ the system (3.4) has a very simple solution in terms of non-deformed Bethe parameters

$$u_k(\bar{\kappa}) = u_k - \frac{i}{L} \log \kappa, \quad v_k(\bar{\kappa}) = v_k - \frac{i}{L} \log \kappa, \quad (3.9)$$

where \bar{u} and \bar{v} are solutions of the standard Bethe equations (2.6). Substituting (3.9) into (3.8) we obtain (3.6).

Theorem 3.3. *The form factors of the Bose fields $\Psi_k(x)$ and $\Psi_k^\dagger(x)$ ($k = 1, 2$) have the following representation:*

$$\mathbb{C}_{a-2+k, b-1}(\bar{u}^C; \bar{v}^C) \Psi_k(x) \mathbb{B}_{a, b}(\bar{u}^B; \bar{v}^B) = i\sqrt{\kappa} e^{ix\mathcal{P}(\bar{v}^B, \bar{v}^C)} \mathfrak{F}_{a, b}^{(3, k)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B), \quad (3.10)$$

$$\mathbb{C}_{a+2-k, b+1}(\bar{u}^C; \bar{v}^C) \Psi_k^\dagger(x) \mathbb{B}_{a, b}(\bar{u}^B; \bar{v}^B) = i\sqrt{\kappa} e^{ix\mathcal{P}(\bar{v}^B, \bar{v}^C)} \mathfrak{F}_{a, b}^{(k, 3)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B), \quad (3.11)$$

where $\mathfrak{F}_{a, b}^{(3, k)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)$ and $\mathfrak{F}_{a, b}^{(k, 3)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)$ are respectively the universal form factors of the matrix elements $T_{3k}(z)$ and $T_{k3}(z)$.

The statement of this theorem cannot be obtained directly from the results of [14]. Here we show that if equation (3.10) holds for $k = 2$, then it is valid for $k = 1$, and then (3.11) is also valid for $k = 1, 2$. The proof of equation (3.10) for $k = 2$ will be given in the next section.

Let us denote form factors of the partial zero modes $T_{ij}^{(1;R)}[0]$ as

$$\mathbb{M}_{a, b}^{i, j}(x) \equiv \mathbb{M}_{a, b}^{i, j}(x | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}_{a', b'}(\bar{u}^C; \bar{v}^C) T_{ij}^{(1;R)}[0] \mathbb{B}_{a, b}(\bar{u}^B; \bar{v}^B). \quad (3.12)$$

Recall that here $a' = a + \delta_{i1} - \delta_{j1}$, $b' = b + \delta_{j3} - \delta_{i3}$. It follows from (2.39) that the form factors of fields $\Psi_k(x)$ can be obtained from the form factors $\mathbb{M}_{a, b}^{3, k}$, $k = 1, 2$. Let us show that $\mathbb{M}_{a, b}^{3, 1}$ and $\mathbb{M}_{a, b}^{3, 2}$ are related to each other by a simple limiting procedure.

Proposition 3.1.

$$\lim_{w \rightarrow +i\infty} \frac{w}{c} \mathbb{M}_{a, b}^{3, 2}(x | \{\bar{u}^C, w\}, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{M}_{a, b}^{3, 1}(x | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B). \quad (3.13)$$

Proof. We have from RTT -relation (2.2)

$$[T_{21}^{(1)}(u), T_{32}^{(1)}(v)] = g(u, v) (T_{31}^{(1)}(v) T_{22}^{(1)}(u) - T_{31}^{(1)}(u) T_{22}^{(1)}(v)). \quad (3.14)$$

Multiplying this equation by $c^2/(uv)$ and sending $u, v \rightarrow +i\infty$ we obtain

$$[T_{21}^{(1)}[0], T_{32}^{(1;R)}[0]] = T_{31}^{(1;R)}[0], \quad (3.15)$$

and thus,

$$[T_{21}[0], T_{32}^{(1;R)}[0]] = T_{31}^{(1;R)}[0], \quad (3.16)$$

because $T_{21}[0] = T_{21}^{(1)}[0] + T_{21}^{(2)}[0]$ and $T_{21}^{(2)}[0]$ commutes with $T_{32}^{(1;R)}[0]$. Due to (2.32) we have

$$\lim_{w \rightarrow +i\infty} \frac{w}{c} \mathbb{M}_{a, b}^{3, 2}(x | \{\bar{u}^C, w\}, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}_{a-1, b-1}(\bar{u}^C; \bar{v}^C) T_{21}[0] T_{32}^{(1;R)}[0] \mathbb{B}_{a, b}(\bar{u}^B; \bar{v}^B). \quad (3.17)$$

Since the action of $T_{21}[0]$ on the on-shell vector $\mathbb{B}_{a, b}(\bar{u}^B; \bar{v}^B)$ gives zero (see (2.33)), we can replace in (3.17) the product $T_{21}[0] T_{32}^{(1;R)}[0]$ by the commutator $[T_{21}[0], T_{32}^{(1;R)}[0]]$. The last one is equal to $T_{31}^{(1;R)}[0]$ due to (3.16). We arrive at

$$\lim_{w \rightarrow +i\infty} \frac{w}{c} \mathbb{M}_{a, b}^{3, 2}(x | \{\bar{u}^C, w\}, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}_{a-1, b-1}(\bar{u}^C; \bar{v}^C) T_{31}^{(1;R)}[0] \mathbb{B}_{a, b}(\bar{u}^B; \bar{v}^B), \quad (3.18)$$

which ends the proof.

It remains to take this limit in (3.10) for $k = 2$. Using (see [15])

$$\lim_{w \rightarrow +i\infty} \frac{w}{c} \mathfrak{F}_{a,b}^{3,2}(\{\bar{u}^C, w\}, \bar{v}^C | \bar{u}^B, \bar{v}^B) = \mathfrak{F}_{a,b}^{3,1}(\bar{u}^C, \bar{v}^C | \bar{u}^B, \bar{v}^B), \quad (3.19)$$

and $\ell_1(w) = 1$ we conclude that if representation (3.10) holds for the form factor of $\Psi_2(x)$, then it holds for the form factor of $\Psi_1(x)$.

Now we should prove that (3.10) yields (3.11). For this we use the mapping (2.20). Due to (2.39) and (3.10) we have

$$\mathbb{M}_{a,b}^{3,k}(x | \bar{u}'^C, \bar{v}'^C; \bar{u}'^B, \bar{v}'^B) = \frac{\ell_3(\bar{v}'^B)}{\ell_3(\bar{v}'^C)} \mathfrak{F}_{a,b}^{(3,k)}(\bar{u}'^C, \bar{v}'^C; \bar{u}'^B, \bar{v}'^B). \quad (3.20)$$

Recall that here the sets $\{\bar{u}'^C, \bar{v}'^C\}$ and $\{\bar{u}'^B, \bar{v}'^B\}$ satisfy the Bethe equations. Replacing in (3.20) x by $L - x$ we obtain

$$\mathbb{M}_{a,b}^{3,k}(L - x | \bar{u}'^C, \bar{v}'^C; \bar{u}'^B, \bar{v}'^B) = \frac{\ell_3(\bar{v}'^C)}{\ell_3(\bar{v}'^B)} \mathfrak{F}_{a,b}^{(3,k)}(\bar{u}'^C, \bar{v}'^C; \bar{u}'^B, \bar{v}'^B). \quad (3.21)$$

Here we have used

$$\frac{\ell_3(\bar{v}'^B)}{\ell_3(\bar{v}'^C)} \Big|_{x \rightarrow L-x} = \frac{r_3(\bar{v}'^B)}{r_3(\bar{v}'^C)} \frac{\ell_3(\bar{v}'^C)}{\ell_3(\bar{v}'^B)} = \frac{\ell_3(\bar{v}'^C)}{\ell_3(\bar{v}'^B)}, \quad (3.22)$$

because due to (2.7) in the TCBG model $r_3(\bar{v}'^B) = r_3(\bar{v}'^C) = 1$. Now we act on (3.21) with the mapping ψ (2.20). The r.h.s. remains invariant, while in the l.h.s. we obtain due to (2.21)

$$\psi \left(\mathbb{M}_{a,b}^{3,k}(L - x | \bar{u}'^C, \bar{v}'^C; \bar{u}'^B, \bar{v}'^B) \right) = -\mathbb{M}_{a,b}^{k,3}(x | \bar{u}'^B, \bar{v}'^B; \bar{u}'^C, \bar{v}'^C). \quad (3.23)$$

Thus, we arrive at

$$\mathbb{M}_{a',b'}^{k,3}(x | \bar{u}'^B, \bar{v}'^B; \bar{u}'^C, \bar{v}'^C) = -\frac{\ell_3(\bar{v}'^C)}{\ell_3(\bar{v}'^B)} \mathfrak{F}_{a,b}^{(3,k)}(\bar{u}'^C, \bar{v}'^C; \bar{u}'^B, \bar{v}'^B). \quad (3.24)$$

Now we simply rename the Bethe parameters

$$\bar{u}'^C \rightarrow \bar{u}^B, \quad \bar{u}'^B \rightarrow \bar{u}^C, \quad \bar{v}'^C \rightarrow \bar{v}^B, \quad \bar{v}'^B \rightarrow \bar{v}^C, \quad \{a', b'\} \leftrightarrow \{a, b\}, \quad (3.25)$$

and use (2.19). We obtain

$$\mathbb{M}_{a,b}^{k,3}(x | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \frac{\ell_3(\bar{v}^B)}{\ell_3(\bar{v}^C)} \mathfrak{F}_{a,b}^{(k,3)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B). \quad (3.26)$$

This equation together with (2.39) implies (3.11).

Thus, in order to prove the statements of theorem 3.3 it is enough to prove (3.10) for $k = 2$. This will be done in the next section.

4 Form factor of the field $\Psi_2(x)$

Due to (2.39) the form factor of the field $\Psi_2(x)$ is related to the form factor $M_{a,b}^{3,2}$ of the right partial zero mode $T_{32}^{(1;R)}[0]$. The latter should be calculated straightforwardly using the action formula (2.40) and the scalar product formula (2.11). We used similar way in [14] for the calculating the form factors of the diagonal partial zero modes. That derivation was based on a summation identity that also plays a very important role in the present case.

4.1 Summation identities

Lemma 4.1. *Let $\bar{u}^{C,B}$ and $\bar{v}^{C,B}$ be four sets of generic complex numbers with cardinalities $\#\bar{u}^{C,B} = a$ and $\#\bar{v}^{C,B} = b$, $a, b = 0, 1, \dots$. For arbitrary partitions of these sets of the form (2.12) define a function W as*

$$W \begin{pmatrix} \bar{u}_I^C, \bar{u}_I^B, & \bar{v}_I^C, \bar{v}_I^B \\ \bar{u}_{II}^C, \bar{u}_{II}^B, & \bar{v}_{II}^C, \bar{v}_{II}^B \end{pmatrix} = f(\bar{u}_{II}^C, \bar{u}_I^C) f(\bar{u}_I^B, \bar{u}_{II}^B) f(\bar{v}_I^C, \bar{v}_{II}^C) f(\bar{v}_{II}^B, \bar{v}_I^B) f(\bar{v}_{II}^C, \bar{u}_{II}^C) f(\bar{v}_I^B, \bar{u}_I^B) \\ \times Z_{a_{II}, b_I}(\bar{u}_{II}^C; \bar{u}_{II}^B | \bar{v}_I^C; \bar{v}_I^B) Z_{a_I, b_{II}}(\bar{u}_I^B; \bar{u}_I^C | \bar{v}_{II}^B; \bar{v}_{II}^C), \quad (4.1)$$

where $Z_{a,b}$ are the highest coefficients (see (2.11)). Then

$$\sum W \begin{pmatrix} \bar{u}_I^C, \bar{u}_I^B, & \bar{v}_I^C, \bar{v}_I^B \\ \bar{u}_{II}^C, \bar{u}_{II}^B, & \bar{v}_{II}^C, \bar{v}_{II}^B \end{pmatrix} = \delta_{0a} \delta_{0b}, \quad (4.2)$$

where the sum is taken over all possible partitions $\bar{u}^{C,B} \Rightarrow \{\bar{u}_I^{C,B}, \bar{u}_{II}^{C,B}\}$ and $\bar{v}^{C,B} \Rightarrow \{\bar{v}_I^{C,B}, \bar{v}_{II}^{C,B}\}$ with $\#\bar{u}_I^C = \#\bar{u}_I^B$ and $\#\bar{v}_I^C = \#\bar{v}_I^B$.

Identity (4.2) was proved in [28]. In order to compute the form factor $M_{a,b}^{3,2}$ we need one more summation identity.

Lemma 4.2. *Let $\bar{u}^{C,B}$ and $\bar{v}^{C,B}$ be as in lemma 4.1 and w be an arbitrary complex number. Let also $\{\bar{v}^C, w\} = \bar{\xi}$. Then*

$$\lim_{|w| \rightarrow \infty} \frac{w}{c} \sum_{w \in \bar{\xi}_I} W \begin{pmatrix} \bar{u}_I^C, \bar{u}_I^B, & \bar{\xi}_I, \bar{v}_I^B \\ \bar{u}_{II}^C, \bar{u}_{II}^B, & \bar{\xi}_{II}, \bar{v}_{II}^B \end{pmatrix} = \mathfrak{F}_{a,b}^{(3,2)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B). \quad (4.3)$$

Here the sum is taken over partitions $\bar{u}^{C,B} \Rightarrow \{\bar{u}_I^{C,B}, \bar{u}_{II}^{C,B}\}$, $\bar{v}^B \Rightarrow \{\bar{v}_I^B, \bar{v}_{II}^B\}$, and $\bar{\xi} \Rightarrow \{\bar{\xi}_I, \bar{\xi}_{II}\}$ with $\#\bar{u}_I^C = \#\bar{u}_I^B$ and $\#\bar{\xi}_I = \#\bar{v}_I^B$, and we demand that $w \in \bar{\xi}_I$. This restriction on the partitions is indicated explicitly by the subscript of the sum.

The proof of this lemma is given in appendix A.

We would like to emphasise that the restriction $w \in \bar{\xi}_I$ is of great importance. Without this restriction the sum in (4.3) vanishes due to lemma 4.1. Therefore, in particular, equation (4.3) implies

$$\lim_{|w| \rightarrow \infty} \frac{w}{c} \sum_{w \in \bar{\xi}_{II}} W \begin{pmatrix} \bar{u}_I^C, \bar{u}_I^B, & \bar{\xi}_I, \bar{v}_I^B \\ \bar{u}_{II}^C, \bar{u}_{II}^B, & \bar{\xi}_{II}, \bar{v}_{II}^B \end{pmatrix} = -\mathfrak{F}_{a,b}^{(3,2)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B), \quad (4.4)$$

where the sum is now taken with the restriction $w \in \bar{\xi}_{II}$. Indeed, adding together the sums in (4.3) and (4.4) gives a sum without any restriction, which is equal to zero.

4.2 Derivation of a determinant representation

Due to (2.40) the action of $T_{32}^{(1;R)}[0]$ on a total dual on-shell Bethe vector is

$$\begin{aligned} \mathbb{C}_{a,b-1}(\bar{u}^C; \bar{v}^C) T_{32}^{(1;R)}[0] &= \lim_{w \rightarrow +i\infty} \frac{w}{c} \sum \frac{\ell_3(\bar{v}_\Pi^C)}{\ell_1(\bar{u}_I^C)} f(\bar{u}_I^C, \bar{u}_\Pi^C) f(\bar{v}_\Pi^C, \bar{v}_I^C) f(\bar{v}_I^C, \bar{u}_I^C) \\ &\quad \times \mathbb{C}_{a_I, b_I}^{(1)}(\bar{u}_I^C; \{w, \bar{v}_I^C\}) \mathbb{C}_{a_\Pi, b_\Pi}^{(2)}(\bar{u}_\Pi^C; \bar{v}_\Pi^C). \end{aligned} \quad (4.5)$$

Let $\{w, \bar{v}^C\} = \bar{\xi}$. Then

$$\begin{aligned} M_{a,b}^{(3,2)}(x) &= \lim_{w \rightarrow +i\infty} e^{iwx} \frac{w}{c} \sum \frac{\ell_1(\bar{u}_\Pi^C) \ell_3(\bar{v}_\Pi^B)}{\ell_1(\bar{u}_I^B) \ell_3(\bar{\xi}_I)} f(\bar{u}_I^C, \bar{u}_\Pi^C) f(\bar{u}_I^B, \bar{u}_\Pi^B) f(\bar{\xi}_\Pi, \bar{\xi}_I) f(\bar{v}_\Pi^B, \bar{v}_I^B) \\ &\quad \times f(\bar{v}_I^B, \bar{u}_I^B) f(\bar{\xi}_I, \bar{u}_I^C) \mathbb{C}_{a_I, b_I+1}^{(1)}(\bar{u}_I^C; \bar{\xi}_I) \mathbb{B}_{a_I, b_I+1}^{(1)}(\bar{u}_I^B; \bar{v}_I^B) \cdot \mathbb{C}_{a_\Pi, b_\Pi}^{(2)}(\bar{u}_\Pi^C; \bar{\xi}_\Pi) \mathbb{B}_{a_\Pi, b_\Pi}^{(2)}(\bar{u}_\Pi^B; \bar{v}_\Pi^B). \end{aligned} \quad (4.6)$$

Here the sum in is taken over the partitions $\bar{u}^{C,B} \Rightarrow \{\bar{u}_I^{C,B}, \bar{u}_\Pi^{C,B}\}$, $\bar{v}^B \Rightarrow \{\bar{v}_I^B, \bar{v}_\Pi^B\}$, and $\bar{\xi} \Rightarrow \{\bar{\xi}_I, \bar{\xi}_\Pi\}$ with a restriction $w \in \bar{\xi}_I$. We also have set $\bar{\xi}_I = \{w, \bar{v}_I^C\}$ and $\bar{\xi}_\Pi = \bar{v}_\Pi^C$.

Since $w \in \bar{\xi}_I$ and $\ell_3(w) = e^{iwx}$, one goes from (4.5) to (4.6) using

$$\ell_3^{-1}(\bar{v}_I^C) = e^{iwx} \ell_3^{-1}(\bar{\xi}_I). \quad (4.7)$$

In doing so, we also replaced the products $f(\bar{v}_\Pi^C, \bar{v}_I^C)$ and $f(\bar{v}_I^C, \bar{u}_I^C)$ by $f(\bar{\xi}_\Pi, \bar{\xi}_I)$ and $f(\bar{\xi}_I, \bar{u}_I^C)$ respectively. This is possible, because if the f function depends on w , then it goes to 1 in the limit $w \rightarrow +i\infty$.

We should substitute the explicit expression for the scalar products (2.11) into (A.7). It is clear that we obtain new partitions of the subsets into subsubsets. Therefore, in order to avoid cumbersome roman numbers, we numerate these subsubsets by the standard arabic numbers.

Using (2.11) for the scalar product of $\mathbb{C}^{(1)}$ and $\mathbb{B}^{(1)}$ we should replace all the functions r_k by ℓ_k :

$$\begin{aligned} \mathbb{C}_{a_I, b_I+1}^{(1)}(\bar{u}_I^C; \bar{\xi}_I) \mathbb{B}_{a_I, b_I+1}^{(1)}(\bar{u}_I^B; \bar{v}_I^B) &= \sum \ell_1(\bar{u}_1^B) \ell_1(\bar{u}_3^C) \ell_3(\bar{\xi}_3) \ell_3(\bar{v}_1^B) f(\bar{u}_1^C, \bar{u}_3^C) f(\bar{u}_3^B, \bar{u}_1^B) \\ &\quad \times f(\bar{\xi}_3, \bar{\xi}_1) f(\bar{v}_1^B, \bar{v}_3^B) \frac{f(\bar{\xi}_1, \bar{u}_1^C) f(\bar{v}_3^B, \bar{u}_3^B)}{f(\bar{\xi}_I, \bar{u}_I^C) f(\bar{v}_I^B, \bar{u}_I^B)} Z_{a_3, b_1}(\bar{u}_3^C; \bar{u}_3^B | \bar{\xi}_1; \bar{v}_1^B) Z_{a_1, b_3}(\bar{u}_1^B; \bar{u}_1^C | \bar{v}_3^B; \bar{\xi}_3). \end{aligned} \quad (4.8)$$

The summation is taken with respect to the partitions

$$\bar{u}_I^{C,B} \Rightarrow \{\bar{u}_1^{C,B}, \bar{u}_3^{C,B}\}, \quad \bar{v}_I^B \Rightarrow \{\bar{v}_1^B, \bar{v}_3^B\}, \quad \bar{\xi}_I \Rightarrow \{\bar{\xi}_1, \bar{\xi}_3\}. \quad (4.9)$$

The cardinalities of the subsubsets are $a_n = \#\bar{u}_n^{C,B}$, $b_n = \#\bar{v}_n^B = \#\bar{\xi}_n$, $n = 1, 3$.

In the scalar product of $\mathbb{C}^{(2)}$ and $\mathbb{B}^{(2)}$ we should replace the functions r_k by r_k/ℓ_k :

$$\begin{aligned} \mathbb{C}_{a_\Pi, b_\Pi}^{(2)}(\bar{u}_\Pi^C; \bar{\xi}_\Pi) \mathbb{B}_{a_\Pi, b_\Pi}^{(2)}(\bar{u}_\Pi^B; \bar{v}_\Pi^B) &= \sum \frac{r_2(\bar{u}_2^B) r_2(\bar{u}_4^C) r_4(\bar{\xi}_4) r_4(\bar{v}_2^B)}{\ell_2(\bar{u}_2^B) \ell_2(\bar{u}_4^C) \ell_4(\bar{\xi}_4) \ell_4(\bar{v}_2^B)} f(\bar{u}_2^C, \bar{u}_4^C) f(\bar{u}_4^B, \bar{u}_2^B) \\ &\quad \times f(\bar{\xi}_4, \bar{\xi}_2) f(\bar{v}_2^B, \bar{v}_4^B) \frac{f(\bar{\xi}_2, \bar{u}_2^C) f(\bar{v}_4^B, \bar{u}_4^B)}{f(\bar{\xi}_\Pi, \bar{u}_\Pi^C) f(\bar{v}_\Pi^B, \bar{u}_\Pi^B)} Z_{a_4, b_2}(\bar{u}_4^C; \bar{u}_4^B | \bar{\xi}_2; \bar{v}_2^B) Z_{a_2, b_4}(\bar{u}_2^B; \bar{u}_2^C | \bar{v}_4^B; \bar{\xi}_4). \end{aligned} \quad (4.10)$$

The summation is taken with respect to the partitions

$$\bar{u}_{\text{II}}^{C,B} \Rightarrow \{\bar{u}_2^{C,B}, \bar{u}_4^{C,B}\}, \quad \bar{v}_{\text{II}}^B \Rightarrow \{\bar{v}_2^B, \bar{v}_4^B\}, \quad \bar{\xi}_{\text{II}} \Rightarrow \{\bar{\xi}_2, \bar{\xi}_4\}. \quad (4.11)$$

The cardinalities of the subsubsets are $a_n = \#\bar{u}_n^{C,B}$, $b_n = \#\bar{v}_n^B = \#\bar{\xi}_n$, $n = 2, 4$.

The next step is to express the functions r_k in (4.10) through the Bethe equations:

$$r_1(\bar{u}_2^B) = \frac{f(\bar{u}_2^B, \bar{u}_1^B)f(\bar{u}_2^B, \bar{u}_3^B)f(\bar{u}_2^B, \bar{u}_4^B)}{f(\bar{u}_1^B, \bar{u}_2^B)f(\bar{u}_3^B, \bar{u}_2^B)f(\bar{u}_4^B, \bar{u}_2^B)} f(\bar{v}^B, \bar{u}_2^B), \quad (4.12)$$

$$r_3(\bar{v}_2^B) = \frac{f(\bar{v}_1^B, \bar{v}_2^B)f(\bar{v}_3^B, \bar{v}_2^B)f(\bar{v}_4^B, \bar{v}_2^B)}{f(\bar{v}_2^B, \bar{v}_1^B)f(\bar{v}_2^B, \bar{v}_3^B)f(\bar{v}_2^B, \bar{v}_4^B)} f(\bar{v}_2^B, \bar{u}^B), \quad (4.13)$$

$$r_1(\bar{u}_4^C) = \frac{f(\bar{u}_4^C, \bar{u}_1^C)f(\bar{u}_4^C, \bar{u}_2^C)f(\bar{u}_4^C, \bar{u}_3^C)}{f(\bar{u}_1^C, \bar{u}_4^C)f(\bar{u}_2^C, \bar{u}_4^C)f(\bar{u}_3^C, \bar{u}_4^C)} f(\bar{\xi}, \bar{u}_4^C), \quad (4.14)$$

$$r_3(\bar{\xi}_4) = \frac{f(\bar{\xi}_1, \bar{\xi}_4)f(\bar{\xi}_2, \bar{\xi}_4)f(\bar{\xi}_3, \bar{\xi}_4)}{f(\bar{\xi}_4, \bar{\xi}_1)f(\bar{\xi}_4, \bar{\xi}_2)f(\bar{\xi}_4, \bar{\xi}_3)} f(\bar{\xi}_4, \bar{u}^C). \quad (4.15)$$

Remark 1. We can explain now why we kept the notation $r_1(u)$ in spite of $r_1(u) = 1$ in the TCBCG model. This function shows explicitly, to which subsubsets we should apply the Bethe equations in (4.10). In our case these are the subsubsets \bar{u}_2^B and \bar{u}_4^C . If we had set $r_1(u) = 1$ in (4.10), then we would have much more freedom and we could use the Bethe equations for other subsubsets, which is inappropriate.

Remark 2. Note that we have used the Bethe equations for the product $r_3(\bar{\xi}_4)$ in spite of the set $\bar{\xi}$ contains the parameter w . In fact, $w \in \bar{\xi}_1$, hence, $w \notin \bar{\xi}_4$, therefore we can use the Bethe equations for the product $r_3(\bar{\xi}_4)$. In the r.h.s. of (4.15) we have w as the argument of the f functions, but this function goes to 1 as $w \rightarrow +i\infty$.

Equations (4.8), (4.10), and (4.12)–(4.15) should be substituted into (A.7). It leads us to the following representation:

$$\mathcal{M}_{a,b}^{(3,2)} = \lim_{w \rightarrow +i\infty} e^{iwx \frac{w}{c}} \sum_{w \in \{\bar{\xi}_1, \bar{\xi}_3\}} \frac{\ell_1(\bar{u}_2^C)\ell_1(\bar{u}_3^C)\ell_3(\bar{v}_1^B)\ell_3(\bar{v}_4^B)}{\ell_1(\bar{u}_2^B)\ell_1(\bar{u}_3^B)\ell_3(\bar{\xi}_1)\ell_3(\bar{\xi}_4)} F_{uu}^C F_{vv}^C F_{vu}^C F_{uu}^B F_{vv}^B F_{vu}^B \mathcal{Z}. \quad (4.16)$$

Here the sum is taken over partitions of every set of Bethe parameters into four subsets

$$\begin{aligned} \bar{u}^{C,B} &\Rightarrow \{\bar{u}_1^{C,B}, \bar{u}_2^{C,B}, \bar{u}_3^{C,B}, \bar{u}_4^{C,B}\}, \\ \bar{\xi} &\Rightarrow \{\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4\}, \\ \bar{v}^B &\Rightarrow \{\bar{v}_1^B, \bar{v}_2^B, \bar{v}_3^B, \bar{v}_4^B\}, \end{aligned} \quad (4.17)$$

with the restriction $w \in \{\bar{\xi}_1, \bar{\xi}_3\}$ that is shown explicitly by the subscript of the sum. The factor \mathcal{Z} in (4.16) is the product of four highest coefficients

$$\mathcal{Z} = Z_{a_3, b_1}(\bar{u}_3^C; \bar{u}_3^B | \bar{\xi}_1; \bar{v}_1^B) Z_{a_1, b_3}(\bar{u}_1^B; \bar{u}_1^C | \bar{v}_3^B; \bar{\xi}_3) Z_{a_4, b_2}(\bar{u}_4^C; \bar{u}_4^B | \bar{\xi}_2; \bar{v}_2^B) Z_{a_2, b_4}(\bar{u}_2^B; \bar{u}_2^C | \bar{v}_4^B; \bar{\xi}_4). \quad (4.18)$$

Other factors in (4.16) denoted by F with different subscripts and superscripts are products of f functions:

$$F_{uu}^C = f(\bar{u}_4^C, \bar{u}_1^C)f(\bar{u}_3^C, \bar{u}_2^C)f(\bar{u}_4^C, \bar{u}_2^C)f(\bar{u}_4^C, \bar{u}_3^C)f(\bar{u}_1^C, \bar{u}_2^C)f(\bar{u}_1^C, \bar{u}_3^C), \quad (4.19)$$

$$F_{uu}^B = f(\bar{u}_1^B, \bar{u}_4^B) f(\bar{u}_2^B, \bar{u}_3^B) f(\bar{u}_2^B, \bar{u}_1^B) f(\bar{u}_2^B, \bar{u}_1^B) f(\bar{u}_3^B, \bar{u}_4^B) f(\bar{u}_3^B, \bar{u}_4^B), \quad (4.20)$$

$$F_{vv}^C = f(\bar{\xi}_1, \bar{\xi}_4) f(\bar{\xi}_2, \bar{\xi}_3) f(\bar{\xi}_2, \bar{\xi}_1) f(\bar{\xi}_2, \bar{\xi}_4) f(\bar{\xi}_3, \bar{\xi}_1) f(\bar{\xi}_3, \bar{\xi}_4), \quad (4.21)$$

$$F_{vv}^B = f(\bar{v}_4^B, \bar{v}_1^B) f(\bar{v}_3^B, \bar{v}_2^B) f(\bar{v}_1^B, \bar{v}_3^B) f(\bar{v}_4^B, \bar{v}_3^B) f(\bar{v}_1^B, \bar{v}_2^B) f(\bar{v}_4^B, \bar{v}_2^B), \quad (4.22)$$

$$F_{vu}^C = f(\bar{\xi}_1, \bar{u}_4^C) f(\bar{\xi}_4, \bar{u}_4^C) f(\bar{\xi}_1, \bar{u}_1^C) f(\bar{\xi}_4, \bar{u}_1^C) f(\bar{\xi}_3, \bar{u}_4^C) f(\bar{\xi}_4, \bar{u}_3^C), \quad (4.23)$$

$$F_{vu}^B = f(\bar{v}_3^B, \bar{u}_3^B) f(\bar{v}_2^B, \bar{u}_2^B) f(\bar{v}_3^B, \bar{u}_2^B) f(\bar{v}_2^B, \bar{u}_3^B) f(\bar{v}_1^B, \bar{u}_2^B) f(\bar{v}_2^B, \bar{u}_1^B). \quad (4.24)$$

It remains to combine the subsubsets into new groups:

$$\begin{aligned} \{\bar{u}_1^{C,B}, \bar{u}_4^{C,B}\} &= \bar{u}_i^{C,B}, & \{\bar{u}_2^{C,B}, \bar{u}_3^{C,B}\} &= \bar{u}_{ii}^{C,B}, \\ \{\bar{\xi}_1, \bar{\xi}_4\} &= \bar{\xi}_i, & \{\bar{\xi}_2, \bar{\xi}_3\} &= \bar{\xi}_{ii}, \\ \{\bar{v}_1^B, \bar{v}_4^B\} &= \bar{v}_i^B, & \{\bar{v}_2^B, \bar{v}_3^B\} &= \bar{v}_{ii}^B. \end{aligned} \quad (4.25)$$

Then we obtain

$$\begin{aligned} M_{a,b}^{(3,2)} &= \lim_{w \rightarrow +i\infty} e^{iwx} \frac{w}{c} \sum \frac{\ell_1(\bar{u}_{ii}^C) \ell_3(\bar{v}_i^B)}{\ell_1(\bar{u}_{ii}^B) \ell_3(\bar{\xi}_i)} f(\bar{u}_i^C, \bar{u}_{ii}^C) f(\bar{u}_{ii}^B, \bar{u}_i^B) f(\bar{\xi}_{ii}, \bar{\xi}_i) f(\bar{v}_i^B, \bar{v}_{ii}^B) \\ &\quad \times f(\bar{\xi}_i, \bar{u}_i^C) f(\bar{v}_{ii}^B, \bar{u}_{ii}^B) G_1(\bar{u}_i^C, \bar{u}_i^B; \bar{\xi}_{ii}, \bar{v}_{ii}^B) G_2(\bar{u}_{ii}^C, \bar{u}_{ii}^B; \bar{\xi}_i, \bar{v}_i^B), \end{aligned} \quad (4.26)$$

where the sum is taken over partitions $\bar{u}^{C,B} \Rightarrow \{\bar{u}_i^{C,B}, \bar{u}_{ii}^{C,B}\}$, $\bar{\xi} \Rightarrow \{\bar{\xi}_i, \bar{\xi}_{ii}\}$, and $\bar{v}^B \Rightarrow \{\bar{v}_i^B, \bar{v}_{ii}^B\}$. The functions G_1 and G_2 in their turn are given as the sums over partitions of the subsets above into subsubsets:

$$G_1(\bar{u}_i^C, \bar{u}_i^B; \bar{\xi}_{ii}, \bar{v}_{ii}^B) = \sum_{w \notin \bar{\xi}_2} W \left(\begin{matrix} \bar{u}_1^C, \bar{u}_1^B, & \bar{\xi}_2, \bar{v}_2^B \\ \bar{u}_4^C, \bar{u}_4^B, & \bar{\xi}_3, \bar{v}_3^B \end{matrix} \right), \quad (4.27)$$

and

$$G_2(\bar{u}_{ii}^C, \bar{u}_{ii}^B; \bar{\xi}_i, \bar{v}_i^B) = \sum_{w \notin \bar{\xi}_4} W \left(\begin{matrix} \bar{u}_2^C, \bar{u}_2^B, & \bar{\xi}_1, \bar{v}_1^B \\ \bar{u}_3^C, \bar{u}_3^B, & \bar{\xi}_4, \bar{v}_4^B \end{matrix} \right), \quad (4.28)$$

where W is defined by (4.1).

In (4.27) and (4.28) we have the sums over partitions with the restrictions indicated explicitly by the subscripts of the sums. These restrictions appear due to the original condition $w \in \bar{\xi}_i$, that implies $w \in \{\bar{\xi}_1, \bar{\xi}_3\}$. It is easy to see, however, that actually one of these sums has no restriction.

Indeed, suppose that $w \in \bar{\xi}_1$. Then the set $\bar{\xi}_{ii} = \{\bar{\xi}_2, \bar{\xi}_3\}$ does not contain the parameter w . Hence, no restriction is imposed on the sum (4.27). Then due to (4.2) we conclude that $G_1 = 0$, unless $\bar{u}_i^{C,B} = \bar{v}_{ii}^B = \bar{\xi}_{ii} = \emptyset$. Then $\bar{u}_{ii}^{C,B} = \bar{u}^{C,B}$, $\bar{v}_i^B = \bar{v}^B$, and $\bar{\xi}_i = \bar{\xi}$.

Similarly, if $w \in \bar{\xi}_3$, then the set $\bar{\xi}_i = \{\bar{\xi}_1, \bar{\xi}_4\}$ does not contain the parameter w , and therefore we have no restrictions in the sum (4.28). Hence, $G_2 = 0$, unless $\bar{u}_{ii}^{C,B} = \bar{v}_i^B = \bar{\xi}_i = \emptyset$. Then $\bar{u}_i^{C,B} = \bar{u}^{C,B}$, $\bar{v}_{ii}^B = \bar{v}^B$, and $\bar{\xi}_{ii} = \bar{\xi}$. We arrive at the following representation

$$M_{a,b}^{(3,2)} = \lim_{w \rightarrow +i\infty} e^{iwx} \frac{w}{c} \left(\Omega_1 \frac{\ell_1(\bar{u}^C) \ell_3(\bar{v}^B)}{\ell_1(\bar{u}^B) \ell_3(\bar{\xi})} + \Omega_2 \right), \quad (4.29)$$

where

$$\Omega_1 = \sum_{w \in \tilde{\xi}_1} W \left(\begin{matrix} \bar{u}_2^C, \bar{u}_2^B, \\ \bar{u}_3^C, \bar{u}_3^B, \end{matrix} \begin{matrix} \bar{\xi}_1, \bar{v}_1^B \\ \bar{\xi}_4, \bar{v}_4^B \end{matrix} \right), \quad \Omega_2 = \sum_{w \in \tilde{\xi}_3} W \left(\begin{matrix} \bar{u}_1^C, \bar{u}_1^B, \\ \bar{u}_4^C, \bar{u}_4^B, \end{matrix} \begin{matrix} \bar{\xi}_2, \bar{v}_2^B \\ \bar{\xi}_3, \bar{v}_3^B \end{matrix} \right). \quad (4.30)$$

Relabeling the subsets we obtain

$$\Omega_1 = \sum_{w \in \tilde{\xi}_I} W \left(\begin{matrix} \bar{u}_I^C, \bar{u}_I^B, \\ \bar{u}_{II}^C, \bar{u}_{II}^B, \end{matrix} \begin{matrix} \bar{\xi}_I, \bar{v}_I^B \\ \bar{\xi}_{II}, \bar{v}_{II}^B \end{matrix} \right), \quad \Omega_2 = \sum_{w \in \tilde{\xi}_{II}} W \left(\begin{matrix} \bar{u}_I^C, \bar{u}_I^B, \\ \bar{u}_{II}^C, \bar{u}_{II}^B, \end{matrix} \begin{matrix} \bar{\xi}_I, \bar{v}_I^B \\ \bar{\xi}_{II}, \bar{v}_{II}^B \end{matrix} \right). \quad (4.31)$$

It is clear that in the limit $w \rightarrow +i\infty$ the coefficients Ω_1 and Ω_2 respectively coincide with (4.3) and (4.4). Indeed, these coefficients are rational functions of their arguments, therefore it is not important how the parameter w approaches infinity. Thus, we obtain

$$M_{a,b}^{(3,2)} = \lim_{w \rightarrow +i\infty} e^{iwx} \left(\frac{\ell_1(\bar{u}^C) \ell_3(\bar{v}^B)}{\ell_1(\bar{u}^B) \ell_3(\bar{\xi})} - 1 \right) \lim_{w \rightarrow +i\infty} \frac{w}{c} \Omega_1 = \frac{\ell_3(\bar{v}^B)}{\ell_3(\bar{v}^C)} \mathfrak{F}_{a,b}^{(3,2)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B), \quad (4.32)$$

where we have used $\ell_1(u) = 1$. Thus, we have calculated the form factor of the right partial zero mode $T_{32}^{(1;R)}[0]$ and using now (2.39) we arrive at (3.10) for $k = 2$.

Conclusion

In this paper we considered form factors of local operators in the TCBG model. We have shown that they can be reduced to the universal form factors of the monodromy matrix entries. The latter were calculated in [15, 25–28], where determinant representations were found. Determinant formulas for form factors allow one to study the problem of correlation functions. It was done already for the model of the one-component Bose gas [31–33]. We hope that the formulas obtained in the present paper will play the same role in studying the TCBG model. Indeed, knowing form factors one can attack the problem of local operators correlation functions. It gives a possibility to compare theoretical predictions with the experimental results obtained for strongly correlated quantum systems (see e.g. [34–38]).

We also have seen that the zero modes method used in [14] for the evaluation of the local operators form factors can be adapted to the case of TCBG model. In spite of these modifications the final results are very similar to the results of [14]. At least the most essential parts of both results are given by the universal form factors of the monodromy matrix entries. It may happen that such universal coefficients also take place for other integrable models, in particular, for the models described by the q -deformed (trigonometric) R -matrix. The study of this question is now in progress.

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A Proof of lemma 4.2

The strategy for the proof of lemma 4.2 is as follows. We consider an auxiliary model solvable by the algebraic Bethe ansatz and possessing the R -matrix (2.1). We present the monodromy matrix of this auxiliary model in the form (2.22) and assume that the total and partial monodromy matrices have the following asymptotic expansions:

$$\begin{aligned} T(u) &= \mathbf{1} + \sum_{n=0}^{\infty} T[n] \left(\frac{c}{u} \right)^{n+1}, \\ T^{(l)}(u) &= \mathbf{1} + \sum_{n=0}^{\infty} T^{(l)}[n] \left(\frac{c}{u} \right)^{n+1}, \quad l = 1, 2. \end{aligned} \quad (\text{A.1})$$

Respectively the total and partial zero modes are defined as

$$\begin{aligned} T_{ij}[0] &= \lim_{|u| \rightarrow \infty} \frac{u}{c} (T_{ij}(u) - \delta_{ij}), \\ T_{ij}^{(l)}[0] &= \lim_{|u| \rightarrow \infty} \frac{u}{c} (T_{ij}^{(l)}(u) - \delta_{ij}), \quad l = 1, 2, \end{aligned} \quad i, j = 1, 2, 3. \quad (\text{A.2})$$

Such a type of models was considered in [14], where the form factors of all partial zero modes were computed. In particular, using the notation of section 3

$$\tilde{\mathbf{M}}_{a,b}^{(3,2)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \left(\frac{\ell_1(\bar{u}^C) \ell_3(\bar{v}^B)}{\ell_1(\bar{u}^B) \ell_3(\bar{v}^C)} - 1 \right) \mathfrak{F}_{a,b}^{(3,2)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B), \quad (\text{A.3})$$

where

$$\tilde{\mathbf{M}}_{a,b}^{(3,2)} \equiv \tilde{\mathbf{M}}_{a,b}^{(3,2)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}_{a,b-1}(\bar{u}^C; \bar{v}^C) T_{32}^{(1)}[0] \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B). \quad (\text{A.4})$$

Recall also that $\mathfrak{F}_{a,b}^{(3,2)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)$ in (A.3) is the universal form factor of the operator $T_{32}(z)$. The main property of the universal form factor is that it does not depend on the model under consideration. Thus, it is the same for all the models possessing the R -matrix (2.1).

Now we should reproduce this result by the straightforward method used in section 4.2. Namely, using the action formula

$$\mathbb{C}_{a,b}^{(1)}(\bar{u}; \bar{v}) T_{32}^{(1)}[0] = \lim_{|w| \rightarrow \infty} \frac{w}{c} \mathbb{C}_{a,b+1}^{(1)}(\bar{u}; \{w, \bar{v}\}), \quad (\text{A.5})$$

and the scalar product (2.11), we will see that form factor (A.4) reduces to the sum (4.3). Comparing the results we obtain the statement of lemma 4.2.

Due to (2.27) and (A.5) the action of $T_{32}^{(1)}[0]$ on a total dual on-shell Bethe vector is

$$\begin{aligned} \mathbb{C}_{a,b-1}(\bar{u}^C; \bar{v}^C) T_{32}^{(1)}[0] &= \lim_{|w| \rightarrow \infty} \frac{w}{c} \sum \frac{\ell_3(\bar{v}_{\text{II}}^C)}{\ell_1(\bar{u}_{\text{I}}^C)} f(\bar{u}_{\text{I}}^C, \bar{u}_{\text{II}}^C) f(\bar{v}_{\text{II}}^C, \bar{v}_{\text{I}}^C) f(\bar{v}_{\text{I}}^C, \bar{u}_{\text{I}}^C) \\ &\quad \times \mathbb{C}_{a_{\text{I}}, b_{\text{I}}}^{(1)}(\bar{u}_{\text{I}}^C; \{w, \bar{v}_{\text{I}}^C\}) \mathbb{C}_{a_{\text{II}}, b_{\text{II}}}^{(2)}(\bar{u}_{\text{II}}^C; \bar{v}_{\text{II}}^C). \end{aligned} \quad (\text{A.6})$$

Let $\{w, \bar{v}^C\} = \bar{\xi}$. Then

$$\begin{aligned} \tilde{M}_{a,b}^{(3,2)} = \lim_{|w| \rightarrow \infty} \frac{w}{c} \sum_{w \in \bar{\xi}_I} \frac{\ell_1(\bar{u}_I^C) \ell_3(\bar{v}_I^B)}{\ell_1(\bar{u}_I^B) \ell_3(\bar{\xi}_I)} f(\bar{u}_I^C, \bar{u}_I^C) f(\bar{u}_I^B, \bar{u}_I^B) f(\bar{\xi}_I, \bar{\xi}_I) f(\bar{v}_I^B, \bar{v}_I^B) \\ \times f(\bar{v}_I^B, \bar{u}_I^B) f(\bar{\xi}_I, \bar{u}_I^C) \mathbb{C}_{a_I, b_I+1}^{(1)}(\bar{u}_I^C; \bar{\xi}_I) \mathbb{B}_{a_I, b_I+1}^{(1)}(\bar{u}_I^B; \bar{v}_I^B) \cdot \mathbb{C}_{a_{II}, b_{II}}^{(2)}(\bar{u}_{II}^C; \bar{\xi}_{II}) \mathbb{B}_{a_{II}, b_{II}}^{(2)}(\bar{u}_{II}^B; \bar{v}_{II}^B). \end{aligned} \quad (\text{A.7})$$

The sum in (A.7) is taken over the partitions $\bar{u}^{C,B} \Rightarrow \{\bar{u}_I^{C,B}, \bar{u}_{II}^{C,B}\}$, $\bar{\xi} \Rightarrow \{\bar{\xi}_I, \bar{\xi}_{II}\}$, and $\bar{v} \Rightarrow \{\bar{v}_I, \bar{v}_{II}\}$ with a restriction $w \in \bar{\xi}_I$. We also have set $\bar{\xi}_I = \{w, \bar{v}_I^C\}$ and $\bar{\xi}_{II} = \bar{v}_{II}^C$.

Comparing (A.7) with (A.6) we see that we replaced the product $\ell_3(\bar{v}_I^C)$ by $\ell_3(\bar{\xi}_I)$. We can do it, because $\ell_3(w) \rightarrow 1$ as $|w| \rightarrow \infty$ due to the asymptotic expansion (A.1). Similarly to (4.6) we also replaced the products $f(\bar{v}_{II}^C, \bar{v}_I^C)$ and $f(\bar{v}_I^C, \bar{u}_I^C)$ by $f(\bar{\xi}_{II}, \bar{\xi}_I)$ and $f(\bar{\xi}_I, \bar{u}_I^C)$, because the f function goes to 1 if one of its arguments goes to infinity.

It remains now to repeat the derivation of section 4.2. All transforms of (A.7) are *exactly the same* as we did for equation (4.6). Therefore we give here only the final result, which is an analog of (4.32)

$$\tilde{M}_{a,b}^{(3,2)} = \lim_{|w| \rightarrow \infty} \left(\frac{\ell_1(\bar{u}^C) \ell_3(\bar{v}^B)}{\ell_1(\bar{u}^B) \ell_3(\bar{\xi})} - 1 \right) \frac{w}{c} \Omega_1, \quad (\text{A.8})$$

where Ω_1 is given by the sum (4.31). In distinction to (4.32) now we have $\ell_3(w) \rightarrow 1$ as $|w| \rightarrow \infty$, and hence,

$$\tilde{M}_{a,b}^{(3,2)} = \left(\frac{\ell_1(\bar{u}^C) \ell_3(\bar{v}^B)}{\ell_1(\bar{u}^B) \ell_3(\bar{v}^C)} - 1 \right) \lim_{|w| \rightarrow \infty} \frac{w}{c} \Omega_1. \quad (\text{A.9})$$

Comparing (A.9) and (A.3) we see that

$$\lim_{|w| \rightarrow \infty} \frac{w}{c} \Omega_1 = \mathfrak{F}_{a,b}^{(3,2)}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B), \quad (\text{A.10})$$

and this is exactly the statement of lemma 4.2.

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